Computer Control Using Optimal Multivariable Feedforward-Feedback Algorithms

This paper presents three methods of designing feedforward compensators which can be combined with multivariable feedback controllers in order to minimize or eliminate errors caused by sustained measurable disturbances. The designs are based on a linear, time-invariant state-space model of the process and minimize a quadratic function of the errors and/or constrain selected steady state offsets to zero. Simulated and experimental data from a computer controlled pilot-plant evaporator show that multivariable feedback-plus-feedforward control is relatively simple to implement, is practical, and gives excellent control.

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SCOPE

The justification for improved control of industrial processes normally lies in some combination of increased throughput, improved efficiency, better product quality, safer operation, and reduced pollution. Improved control is generally obtained by adding additional conventional controllers or using more sophisticated designs. Most companies have sufficient experience with single variable control systems so that the question of determining the optimum number is a relatively straightforward task of engineering design and justification. However, the use of more sophisticated multivariable controllers has generally lagged because of the lack of convenient design procedures, difficulties regarding implementation, or lack of convincing demonstrations that the theoretical methods will work on real applications. Fortunately progress is being made in overcoming all these difficulties.

In particular, industry is installing increasing numbers of real-time digital control computers which essentially eliminate the physical restrictions formerly imposed by lack of suitable control hardware. Furthermore, since the central processing unit (CPU) utilization on many of these installations is often as low as 5%, the incremental cost of replacing a system of conventional single variable control algorithms where, for example,

$$u(t) = K_{FB}x(t) \quad K_{FB} = \text{diagonal} \qquad (1)$$

by more sophisticated multivariable controllers is often negligible in terms of equipment required.

The work described in this paper is part of a continuing series of projects at the University of Alberta directed towards developing control techniques that are of interest to industry and evaluating them by experimental implementation on computer controlled pilot plants. Because of their relative simplicity and practicality, attention was first directed towards developing multivariable control laws similar to Equation (1) but without the restriction that K_{FB} be diagonal. Several methods for determining such a K_{FB} are referenced later but the method employed in this project was based on a linear, discrete, time-invariant state-space model of the process and a quadratic performance index summed over an infinite interval. The well-known multivariable proportional feedback control law which results from this approach was extended by Newell and Fisher (1972a) to include integral action and provision for making step changes in setpoints or applying model following techniques. This paper describes and evaluates the addition of multivariable feedforward control.

Feedforward control is of considerable interest because, in contrast to feedback control, it can start to compensate for process disturbances before any change occurs in the controlled variables. In certain situations feedforward controllers can completely compensate for disturbances and hence produce perfect regulatory control. However, in practice feedforward controllers are usually combined with feedback action which compensates for modelling errors, controller simplifications, unmeasured disturbances, etc.

Experimental studies by Jacobson (1970) and Newell (1971) showed that multivariable proportional control gave excellent results when applied to a pilot plant evaporator. However, small offsets (deviations) occurred between the actual and desired values of the process variables when constant disturbances were applied to the process. These offsets could be eliminated by integral action, but this increased the dimension of the state vector and the resulting controller. In this study feedforward control is added to the feedback controller as a means of minimizing a quadratic function of the deviations produced by constant, measurable disturbances and/or eliminating a subset of the steady state offsets.

CONCLUSIONS AND SIGNIFICANCE

This work and related projects show that multivariable optimal control techniques are simple and practical to implement and, when applied to a pilot plant evaporator, gave much better control than was ever achieved using conventional, single variable controllers. The figures in this paper reproduce data taken directly from the process by the computer and show that the concentration of the

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product stream from the evaporator was essentially constant in spite of sustained disturbances in feed flow (20%), feed temperature (16%), or feed concentration (30%). Typical responses of the evaporator product concentration to similar disturbances when the evaporator was controlled by conventional single variable controllers, show maximum deviations of over 20% (of mean value) and oscillations that persist for more than two hours.

More specifically it was shown that for a system with m inputs subjected to constant measurable disturbances, feedforward control could be combined with a multivariable feedback controller to:

1. eliminate steady state offsets in up to m variables, and/or
2. minimize the steady state offsets in some subset of the state variables, or
3. minimize a summed quadratic function of the error between the actual and desired values of the system state variables.

It was shown that all three of the above design objectives, plus some of the methods presented in the literature, could be described by a single, generalized interpretation of the optimal control problem for linear, discrete time-invariant processes with quadratic performance indices. This same approach can be used to generate a multivariable control law containing feedback, feedforward, integral, and model following options, and therefore includes all the common options familiar to users of single variable controllers.

The simulation results confirmed that the feedforward controllers meet their design objectives. In the experimental runs the improvement due to adding feedforward control was significant, but process noise made it impossible to distinguish between the different feedforward designs. It should be noted that the controllers were designed on the basis of a linear model which contained significant modelling errors, were implemented using only the simplest filtering and state estimation techniques, and were operated in conjunction with an industrial type direct-digital-control (DDC) system, plus other applications, on a multiprogrammed real-time digital control computer. In spite of these factors they gave excellent control and hence would appear to be robust and practical.

PREVIOUS WORK

During the last decade the design of feedforward controllers for use with conventional single variable control systems has received considerable attention. Miller et al. (1969) in their tutorial article summarize most of the important results and design techniques available at that time. Later investigations include both theoretical and experimental work. Bertran and Chang (1970) studied optimal feedforward control of nonlinear chemical reactors and included distributed parameter terms in their model. Paraskos and McAvoy (1970) used a finite difference approach to develop both transient and steady state feedforward compensation for a class of distributed parameter processes and successfully applied it in experimental studies of a heat exchanger. Feedforward control has also been applied to a laboratory heat exchanger by Corlis and Luus (1969, 1970) and to a stirred tank reactor by West and McGuire (1969b).

By contrast, multivariable feedforward control has received relatively little attention and experimental verification of proposed multivariable control systems has seldom been reported. The transfer function or frequency domain approach was investigated by Bollinger and Lamb (1965), Foster and Stevens (1967), and Luyben (1969). When working with mathematical models it is possible, as shown by Haskins and Sliepcevich (1965) and others, to compensate exactly for load disturbances. However, the required controllers are often complex, sometimes physically unrealizable, and are usually only approximated in physical applications. Greenfield and Ward (1968) also studied feedforward control and point out that the transfer function approach usually requires that the number of control variables must equal the number of outputs. Optimal feedforward-feedback controllers have also been designed using linear state-space process models and minimizing a quadratic performance index of state and control variables. West and McGuire (1969b) used continuous process models, Anderson (1969) used a discrete model, and Solheim and Saethre (1968) considered both types of models. Stochastic disturbances have also been considered by Clifton and McGuire (1971). Heidemann and Esterson (1967) used feedforward action based on the steady state model to reduce offsets. Anderson (1969) presented a design technique based on an error coordinate transformation. Assuming constant disturbances, the system was transformed into the standard optimal regulator problem without disturbances, which could then be solved by existing techniques. The resulting control law, when transformed out of error coordinates, included feedforward action which essentially ensured that selected outputs were at their desired values when the process reached steady state. Latour (1971) has recently shown that for constant disturbances multivariable proportional-integral control is optimal for a quadratic performance index which penalizes the rate of change of the control vector and, of course, eliminates steady state offsets.

Johnson (1968, 1970a, b) has considered the design of disturbance-absorbing controllers. This design approach is based on optimal control theory and seeks to counteract the effects of disturbances by generating a control action which produces a compensating change in the state variables. The disturbance need not be measured but must satisfy a linear differential equation. Sobral and Stefanek (1970) have demonstrated that for certain classes of disturbances, the optimal control formulation reduces to the well-known formulation for systems without disturbances.

This paper considers the problem of designing feedforward controllers that can be added to existing multivariable feedback systems to minimize the effects of measurable, sustained disturbances.

DESIGN BASIS

Consider the standard state-space model of the form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) + \mathbf{D}\,\mathbf{d}(t) \quad \mathbf{x}(0) = \mathbf{x}_0 \qquad (2)$$

$$\mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t)$$

Several of the design techniques mentioned in the previous sections can be used to generate a multivariable control system for a process defined by these equations. In most cases a suitable control law can be formulated in terms of either the r output variables $\mathbf{y}$ or the n state variables $\mathbf{x}$. In this paper the formulation will be in terms of state variables because the optimal control design technique leads naturally to state feedback and because in the evaporator application, to be presented later, the output variables are simply a subset of the state variables. The equivalent formulations in terms of output variables can usually be derived by following the same procedures, and notes to this effect are included in the following sections. Also for simplicity it is assumed that $\mathbf{x}_0 = \mathbf{x}_d = \mathbf{0}$. The case where $\mathbf{x}_d$ is nonzero has been treated by Newell and Fisher (1972a).

Assume that a basic feedback control law with the form given by Equation (3) has already been developed

$$\mathbf{u}(t) = \mathbf{K}_{FB}\, \mathbf{x}(t) \tag{3}$$

This control law will frequently give satisfactory control when used alone. However, consider the case where constant sustained disturbances are applied to a process. If the process is described by Equation (2) and the control action by Equation (3), then when the process finally comes to steady state, $\dot{\mathbf{x}} = \mathbf{0}$ and

$$\mathbf{x}_s = -\,(\mathbf{A} + \mathbf{B}\,\mathbf{K}_{FB})^{-1}\,\mathbf{D}\,\mathbf{d} \tag{4}$$

Since the desired value of the state is zero in this formulation of the regulator control problem, it is obvious from Equation (4)that constant disturbances will cause an offset in $\mathbf{x}$. In many applications such offsets are undesirable and a design procedure that would eliminate and/or minimize such offsets would be advantageous. Also it is reasonable to expect that if the disturbances are measurable then control could be improved over the entire time period of the transient by adding a feedforward term to Equation (3). The following discussion indicates one way of achieving these design objectives.

Assume that zero offsets are required in q of the state variables. The number of degrees of freedom in a control system is limited by the number of control variables so that in general $q \leq m$. It is then convenient to partition $\mathbf{x}$ and $\mathbf{u}$ as follows

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} \tag{5}$$

where $\mathbf{x}_1$ and $\mathbf{u}_1$ both have dimension q. Assuming that it is desired to utilize all m elements of $\mathbf{u}$ in the feedback control action implied by Equation (3), then a generalized design procedure can be stated as:

1. Design the $m \times n$ feedback control matrix $\mathbf{K}_{FB}$
2. Utilize $\mathbf{u}_1$ to produce zero offsets in the q elements of $\mathbf{x}_1$ (note that the case where $q = 0$ is allowed)
3. Utilize $\mathbf{u}_2$ to minimize the steady state offsets in some or all of the remaining $(n - q)$ state variables.

Alternatively, in place of the above three steps:

4. Utilize $\mathbf{u}$ to minimize a quadratic function of the error $(\mathbf{x} - \mathbf{x}_d)$ over the entire transient and hence define the feedforward and feedback action simultaneously. (Note, if a subset of $\mathbf{u}$, say $\mathbf{u}_2$ is used in this step then the feedback action will only include $\mathbf{u}_2$ and this is generally not desirable.)

The method used to design the feelback control matrix $\mathbf{K}_{FB}$ in step 1 is arbitrary and could, for example, be based on standard optimal control techniques as presented in textbooks, for example, Ogata (1967) or Lapidus and Luus (1967); design methods for noninteraction, for example, Gilbert (1969); frequency domain design techniques, for example, Rosenbrock (1970); or eigenvalue assignment techniques, for example, Porter (1969) or Wonham (1967). The optimal control approach is illustrated in a later section.

DESIGN TECHNIQUES FOR FEEDFORWARD CONTROLLERS

The following subsections will follow the general design technique presented in the previous section and develop the design equations for three feedforward controllers, each of which has a different design objective. The designs are all based on a discrete state-space model of the process. If $\mathbf{u}$ and $\mathbf{d}$ are constant during each sampling interval, $iT \leq t \leq (i+1)T$, then the discrete model defined by Equation (6) is equivalent at the sampling instants to the continuous model defined by Equation (2):

$$\mathbf{x}((i+1)T) = \boldsymbol{\varphi}\mathbf{x}(iT) + \boldsymbol{\Delta}\mathbf{u}(iT) + \boldsymbol{\theta}\mathbf{d}(iT), \qquad i = 0, 1, 2, \ldots \tag{6}$$

$$\mathbf{y}(iT) = \mathbf{C}\,\mathbf{x}(iT)$$

where the constant coefficient matrices $\boldsymbol{\varphi}$, $\boldsymbol{\Delta}$, and $\boldsymbol{\theta}$ are evaluated from

$$\boldsymbol{\varphi} = \int_0^T e^{\mathbf{A}t}\,dt \tag{7}$$

$$\boldsymbol{\Delta} = \int_0^T e^{\mathbf{A}(T-t)}\,\mathbf{B}dt \tag{8}$$

$$\boldsymbol{\theta} = \int_0^T e^{\mathbf{A}(T-t)}\,\mathbf{D}dt \tag{9}$$

Design for Zero Steady State Offsets

The first objective, zero offset in the q elements of $\mathbf{x}$ which make up $\mathbf{x}_1$, can be achieved as follows. Assume that the desired control law has the form

$$\mathbf{u}(iT) = \mathbf{K}_{FB}\,\mathbf{x}(iT) + \mathbf{u}_{FF}(iT) \tag{10}$$

and that $\mathbf{K}_{FB}$ has already been determined. The feedforward control action $\mathbf{u}_{FF}$ which will compensate for the steady state offsets caused by constant disturbances, can then be calculated. Substitution of Equation (10) into Equation (6) gives the closed-loop system response

$$\mathbf{x}((i+1)T) = (\boldsymbol{\varphi} + \boldsymbol{\Delta}\,\mathbf{K}_{FB})\,\mathbf{x}(iT) + \boldsymbol{\Delta}\,\mathbf{u}_{FF}(iT) + \boldsymbol{\theta}\,\mathbf{d}(iT) \tag{11}$$

If the system is at steady state such that $\mathbf{x}((i+1)T) = \mathbf{x}(iT)$ and $\mathbf{T}$ is defined by

$$\mathbf{T} = (\boldsymbol{\varphi} + \boldsymbol{\Delta}\,\mathbf{K}_{FB}) - \mathbf{I} \tag{12}$$

then Equation (11) can be written in partitioned form as defined in Equation (5) to yield

$$\mathbf{0} = [\mathbf{T}_1 \,\vdots\, \mathbf{T}_2] \begin{bmatrix} \mathbf{x}_1(iT) \\ \mathbf{x}_2(iT) \end{bmatrix} + [\boldsymbol{\Delta}_1, \boldsymbol{\Delta}_2] \begin{bmatrix} \mathbf{u}_{1FF}(iT) \\ \mathbf{u}_{2FF}(iT) \end{bmatrix} + \boldsymbol{\theta}\,\mathbf{d} \tag{13}$$

If $\mathbf{x}_1(iT)$ is constrained to equal the desired values $\mathbf{x}_{1d}$, then Equation (13) can be rearranged to

$$[\boldsymbol{\Delta}_1 \,\vdots\, \mathbf{T}_2] \begin{bmatrix} \mathbf{u}_{1FF} \\ \mathbf{x}_2 \end{bmatrix} = -\,\boldsymbol{\Delta}_2\,\mathbf{u}_{2FF} - \mathbf{T}_1\,\mathbf{x}_{1d} - \boldsymbol{\theta}\,\mathbf{d} \tag{14}$$

Solving Equation (14) for $\mathbf{u}_{1FF}$ and $\mathbf{x}_2$ gives an expression which can be partitioned and put in the form

$$\mathbf{u}_{1FF}(iT) = \mathbf{K}_1\,\mathbf{u}_{2FF}(iT) + \mathbf{K}_2\,\mathbf{x}_{1d}(iT) + \mathbf{K}_3\,\mathbf{d} \tag{15}$$

In the above derivation it has been assumed that $[\boldsymbol{\Delta}_1, \mathbf{T}_2]^{-1}$ exists and that the disturbance vector $\mathbf{d}$ is constant. However, in practice the resulting feedforward-feedback control law in Equation (15) can be used when the disturbances vary with time providing that the disturbances are measurable. If Equation (15) is substituted into Equation (11), the dynamic response of the controlled system is given by

$$\mathbf{x}((i+1)T) = (\boldsymbol{\varphi} + \boldsymbol{\Delta}\,\mathbf{K}_{FB})\,\mathbf{x}(iT) + (\boldsymbol{\Delta}_1\,\mathbf{K}_1 + \boldsymbol{\Delta}_2)\,\mathbf{u}_{2FF}(iT) + (\boldsymbol{\Delta}_1\,\mathbf{K}_3 + \boldsymbol{\theta})\,\mathbf{d}(iT) + \boldsymbol{\Delta}_1\,\mathbf{K}_2\,\mathbf{x}_{1d} \quad (16)$$

To complete the control system design, $\mathbf{u}_{2FF}$ must be specified. As indicated in the general design procedure, if the dimension of $\mathbf{u}_{2FF}$ is nonzero it can be selected to minimize a function of the offset in $\mathbf{x}_2$. This approach will be examined in the next section.

Thus the control law given in Equation (15) will generate zero offsets in the q state variables forming $\mathbf{x}_1$. It has been shown by Newell (1971) that the "error coordinate" approach of Anderson (1969) is equivalent to the control law given by Equation (15) for the special case where $q = m$ and $\mathbf{K}_{FB}$ is selected to be the optimal feedback matrix obtained using a quadratic performance index.

Minimization of Steady State Offsets

An expression for $\mathbf{u}_{FF}$ in Equation (10) or $\mathbf{u}_{2FF}$ in Equation (15) can be obtained by minimizing a weighted sum-of-the-squared-offsets resulting from a disturbance. Define the performance index J as

$$J = \mathbf{e}_s^T\,\mathbf{S}\,\mathbf{e}_s \quad (17)$$

where the steady state error $\mathbf{e}_s$ is defined as $[\mathbf{x}(NT) - \mathbf{x}_d]$. The following expression for $\mathbf{x}(NT)$ is easily derived from Equations (10) and (11) if it is assumed that the disturbance $\mathbf{d}$ is constant

$$\mathbf{x}(NT) = +\,(\mathbf{I} - \boldsymbol{\varphi} - \boldsymbol{\Delta}\,\mathbf{K}_{FB})^{-1}\,(\boldsymbol{\Delta}\,\mathbf{u}_{FF} + \boldsymbol{\theta}\,\mathbf{d}) \quad (18)$$

To determine the value of $\mathbf{u}_{FF}$ which minimizes J set

$$\frac{\partial J}{\partial \mathbf{u}_{FF}} = \frac{\partial}{\partial \mathbf{u}_{FF}}\,(\mathbf{e}^T\,\mathbf{S}\,\mathbf{e}) = \mathbf{0} \quad (19)$$

Solving for $\mathbf{u}_{FF}$ for the case where $\mathbf{x}_d = \mathbf{0}$ gives

$$\mathbf{u}_{FF} = -\,[(\boldsymbol{\Delta}^T\,\boldsymbol{\varphi}^{*T}\,\mathbf{S}\,\boldsymbol{\varphi}^*\,\boldsymbol{\Delta})^{-1}\,\boldsymbol{\Delta}^T\,\boldsymbol{\varphi}^{*T}\,\mathbf{S}\,\boldsymbol{\varphi}^*\,\boldsymbol{\theta}]\,\mathbf{d} \quad (20)$$

where

$$\boldsymbol{\varphi}^* = +\,(\mathbf{I} - \boldsymbol{\varphi} - \boldsymbol{\Delta}\,\mathbf{K}_{FB})^{-1} \quad (21)$$

For the special case where the state and control vectors have the same dimension (that is, $n = m$) and $\boldsymbol{\Delta}$ is nonsingular then Equation (20) reduces to

$$\mathbf{u}_{FF} = -\,\boldsymbol{\Delta}^{-1}\,\boldsymbol{\theta}\,\mathbf{d} \quad (22)$$

For this special case it is obvious from an inspection of Equation (6) or (18) that $\mathbf{u}_{FF}$ gives perfect dynamic compensation for the effect of the disturbances as well as producing zero offsets.

Optimal Feedforward Plus Feedback Controller

The optimal control formulation for determining both the feedforward and feedback control matrices has been studied by Solheim and Saethre (1968), West and McGuire (1969b), and Sobral and Stefanek (1970). In the discrete formulation the process is modeled by a state-space equation in the form of Equation (6) and the quadratic performance index to be minimized is normally of the form

$$J = \mathbf{e}^T(NT)\,\mathbf{S}\,\mathbf{e}(NT) + \sum_{i=1}^{N} \mathbf{e}^T(iT)\,\mathbf{Q}\,\mathbf{e}(iT) + \mathbf{u}^T((i-1)T)\,\mathbf{R}\,\mathbf{u}((i-1)T) \quad (23)$$

Assuming that the disturbances are constant over the interval $0 \leq t \leq NT$ and that the desired operating point is at the origin (that is, that $\mathbf{x}_d = \mathbf{0}$), then the application of discrete dynamic programming techniques results in a control law of the form:

$$\mathbf{u}(iT) = \mathbf{K}_{FB}\,\mathbf{x}(iT) + \mathbf{K}_{FF}\,\mathbf{d}(iT) \quad (24)$$

where the optimal control matrices are defined by the recursive relations given in the Notation section. These recursive relations were found to converge quickly to produce constant control matrices and for the case where the disturbances are neglected (that is, where $\mathbf{d} = \mathbf{0}$), then $\mathbf{K}_{FB}$ is the optimal feedback control matrix. The case where the desired values $\mathbf{x}_d$ (or $\mathbf{y}_d$) are nonzero has been treated by Newell and Fisher (1972a).

A performance index of the same form as Equation (23) can be defined with the error defined in terms of the output variables, (that is, with $\mathbf{e} = \mathbf{y} - \mathbf{y}_d$): However, if $\mathbf{y} = \mathbf{C}\,\mathbf{x}$ then this formulation is the same as that given by Equation (23) with $\mathbf{S}$ and $\mathbf{Q}$ replaced by $\mathbf{C}^T\mathbf{S}\mathbf{C}$ and $\mathbf{C}^T\mathbf{Q}\mathbf{C}$ respectively and the same control law applies. Also it is obvious that if $\mathbf{Q}$ and $\mathbf{R}$ are null matrices that the performance index given by Equation (23) reduces to that given by Equation (17). Thus the design of a feedforward controller to minimize the steady state offsets can be treated as a special case of this more general problem.

Similarly the specification of zero steady state offsets could be considered as simply a constraint on the final values of q of the state variables. Thus all three design objectives treated in this paper could be covered by an optimal control formulation similar to that presented in this section. However, the recursive relations given in this paper for $\mathbf{K}_{FB}$ and $\mathbf{K}_{FF}$ do not include provision for constraints on $\mathbf{x}$, $\mathbf{y}$, or $\mathbf{u}$.

SIMULATED AND EXPERIMENTAL APPLICATIONS

The following four design objectives, which can all be obtained by adding feedforward control to a multivariable feedback system, were examined in this study.

1. design for zero offsets in $W1$, $W2$, and $C2$
2. design for a zero offset in $C2$ and to minimize offsets in $W1$ and $W2$
3. design to minimize offsets in $W1$, $W2$ and $C2$
4. design both $\mathbf{K}_{FF}$ and $\mathbf{K}_{FB}$ to minimize the quadratic performance index given by Equation (23).

The feedback control matrix used in all four cases was evaluated from the recursive relationships in the notation section with the following parameters:

$$\mathbf{Q} = \text{diag}\,(10,\ 1,\ 1,\ 10,\ 100) \qquad \mathbf{R} = \mathbf{0},\quad \mathbf{S} = \mathbf{0},\quad T = 64 \text{ sec.} \quad (25)$$

Previous work by the authors indicated that these values were a satisfactory combination for this application. The feedforward control matrices evaluated in the four cases are very similar and are shown in Table 1 for cases 1 and 4.

The simulated and experimental applications of the feedforward controllers were carried out using a state-

TABLE 1. CONTROL MATRICES

$$\mathbf{K}_{FB} = \begin{bmatrix} 5.095 & -1.475 & -2.68 & 0 & -14.56 \\ 3.95 & 0.36 & 0.21 & 0 & 7.39 \\ 5.31 & 1.19 & -0.11 & 15.83 & 18.81 \end{bmatrix}$$

$\mathbf{K}_{FF}$ Case 1

$$\begin{bmatrix} 2.055 & -0.090 & -0.463 \\ 1.014 & 0.032 & 0 \\ 1.122 & 0.091 & 0 \end{bmatrix}$$

$\mathbf{K}_{FF}$ Case 4

$$\begin{bmatrix} 2.047 & -0.136 & -0.463 \\ 1.019 & 0.037 & 0 \\ 1.135 & 0.116 & 0 \end{bmatrix}$$

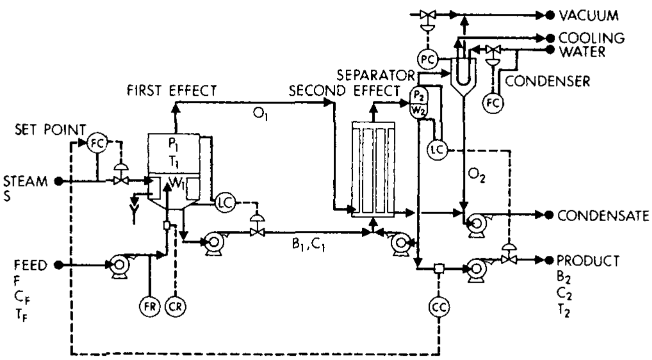

Fig. 1. Schematic diagram of the double effect, computer controlled, pilot plant evaporator that was used as the basis for all simulated and experimental work. The instrumentation is shown connected as conventional single variable controllers.

space model of a pilot-plant, double-effect evaporator in the Department of Chemical and Petroleum Engineering at the University of Alberta, which is described in the next section.

Double Effect Evaporator and Model

A schematic flow diagram of the evaporator in a double effect "forward feed" mode of operation is shown in Figure 1. The first effect is a calandria type unit with an 8-in. diam. tube bundle. It operates with a nominal feedrate of 2.27 kg/min of 3% aqueous triethylene glycol. The second effect is a long-tube-vertical unit with three 0.0254 × 1.83m tubes and is operated with externally forced circulation. The second effect is operated under vacuum and utilizes the vapor from the first effect as the heating medium. The product is about 10% glycol when the steam to the first effect is at its nominal flowrate of 0.9 kg./min.

The state-space model of the evaporator is fifth order and was obtained by linearizing material and energy balances. The process variables in the model are in the normalized perturbation form, for example $(x_i - x_{is})/x_{is}$. The model equations are presented in full in Equation (26). The output vector y is defined to be a subset of the state vector, namely the holdups in the first and second effects $W1$ and $W2$ and the product concentration $C2$. Normal steady state values of the process variable are also given in the notation section.

The multivariable control systems were implemented using an IBM 1800 digital control computer which is interfaced with the pilot-plant double-effect evaporator. The process runs under Direct Digital Control (DDC) and a time-sharing executive system which permits simultaneous execution of off-line jobs.

Multivariable control calculations are carried out by a queued Fortran program which executes every control interval. The basic difference in the computations required for the multivariable versus single variable proportional control algorithms to control the evaporator was 15 multiplications instead of 3. However, the program also does some extra operations such as state estimation and storing desired data on disk. It obtains state variable measurements from DDC data acquisition loops and makes control variable changes by adjusting the set-points of DDC flow control loops.

Additional information on the evaporator, the model, the design procedure, and the implementation are available in the thesis by Newell (1971).

$$\begin{bmatrix} \dot{W}1 \\ \dot{C}1 \\ \dot{H}1 \\ \dot{W}2 \\ \dot{C}2 \end{bmatrix} = \begin{bmatrix} 0 & -.00156 & -.1711 & 0 & 0 \\ 0 & -.1419 & .1711 & 0 & 0 \\ 0 & -.00875 & -1.102 & 0 & 0 \\ 0 & -.00128 & -.1489 & 0 & .00013 \\ 0 & .0605 & .1489 & 0 & -.0591 \end{bmatrix} \begin{bmatrix} W1 \\ C1 \\ H1 \\ W2 \\ C2 \end{bmatrix} + \begin{bmatrix} 0 & -.143 & 0 \\ 0 & 0 & 0 \\ .392 & 0 & 0 \\ 0 & .108 & -.0592 \\ 0 & -.0486 & 0 \end{bmatrix} \begin{bmatrix} S \\ B1 \\ B2 \end{bmatrix} + \begin{bmatrix} .2174 & 0 & 0 \\ -.074 & .1434 & 0 \\ -.036 & 0 & .1814 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} F \\ CF \\ HF \end{bmatrix} \quad (26)$$

Simulation Results

Simulated results are exemplified by Figure 2 where multivariable feedforward-feedback control (case 4) is compared to multivariable feedback alone for a 10% increase in feed flow rate. The oscillations in $W2$ resulted from simulated roundoff in the measured values of the state variables and the expanded scale on Figure 2. However, it will be noted that in the run with feedforward action, the response of the input variables $B1$ and S was faster and stronger than in the run using only feedback control with the result that the maximum deviations in the controlled variables $C2$, $W1$, and $W2$ were smaller and the steady state offsets negligible.

Comparative results for the four cases are also shown in Table 2 for a 50% increase in feed flow rate. In all cases the response exhibited a very small overshoot in the initial transient. As would be expected, the steady state design technique for zero offset (case 1) gave the smallest average offset, while those for the minimization techniques were larger.

Practically speaking, the resulting feedforward control matrices and transient responses were very similar in the four cases examined. However, the addition of feedforward action resulted in a significant increase in control quality over the feedback only case as illustrated by the results in Figure 2. The zero offset design technique involved the least computation although the dynamic programming design can be carried out in conjunction with the evaluation of the proportional feedback matrix.

A simulation study using a distillation column model was carried out by Newell (1971) and led to essentially the same conclusions.

Table 2. Comparison of Control Systems*
(50% Step Increase in Feed Flow Rate)

Case	W1 Offset	$W1_{max}$	W2 Offset	$W2_{max}$	C2 Offset	$C2_{max}$
Feedback	15.5	15.5	0.11	0.25	−1.45	−1.45
FF-FB (1)	0.0002	1.22	0.0009	0.002	−0.0002	−0.38
FF-FB (2)	0.015	1.22	0.0069	0.014	−0.0093	−0.38
FF-FB (3)	0.00003	1.19	0.00001	0.00002	−0.031	−0.40
FF-FB (4)	0.014	1.19	10^{-5}	10^{-5}	−0.028	−0.40

* Note: Offsets and maximum values are expressed as percentage of steady state value.

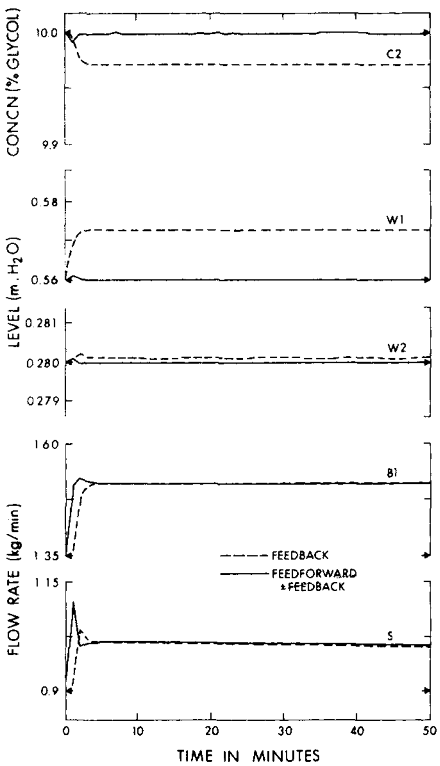

Fig. 2. Comparison of simulated evaporator responses, for a 10% increase in feed flow, using multivariable feedback and feedback-plus-feedforward controllers (case 4). The arrowheads on the vertical axis define the initial steady state of the evaporator.

variables in Figures 4 to 7, was eventually traced to a noisy differential pressure transmitter on the first effect liquid level, that is, $W1$. This noise produced a band on the recorder of about 2% of the span (about 0.6 in. of

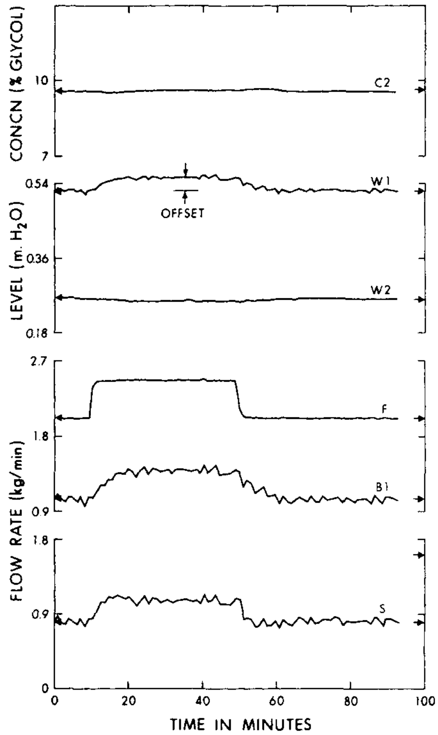

Fig. 3. Experimental response to two 20% changes in feedflow using optimal feedback control only. Comparison with Figures 4 and 7 shows the effect of adding feedforward control.

Experimental Results

The offsets which result from load changes when multivariable proportional feedback control is implemented are exemplified by the first effect holdup ($W1$) curve in Figure 3. The two 20% feed flowrate changes, an increase and then a decrease, produced significant offsets in first effect holdup ($W1$) and very small offsets in product concentration ($C2$) and second effect holdup ($W2$). However, the essentially constant product concentration $C2$ represents a dramatic improvement over the best responses obtained from conventional single variable control using experimentally tuned DDC algorithms. Results presented by Jacobson (1970) and Fisher and Jacobson (1971) showed that a typical response to a 20% feed disturbance when the evaporator was controlled by the conventional single variable controllers shown schematically in Figure 1 involved a maximum percentage deviation in $C2$ of approximately 20% and oscillations that persisted for over two hours. [A typical concentration response is also included in the publications by Newell and Fisher (1972a, b).]

The addition of feedforward control (case 4) was found to reduce the offsets due to load changes in feed flow rate (Figure 4), feed concentration (Figure 5), and feed temperature (Figure 6). When feedforward control is used, noise in the controlled inputs **u** could also result from noisy measurements of the disturbances **d**. However, as shown for example by the recording of the flow F in Figure 4, the disturbances in this set of experiments were essentially noise free and hence unlikely to be the cause of the increased variations in $B1$ and S. Similarly, since $\mathbf{K}_{FB}$ is independent of $\mathbf{K}_{FF}$ and **d**, it was the same for all runs and hence does not affect the comparison. The increased noise in first effect level, and hence in the control

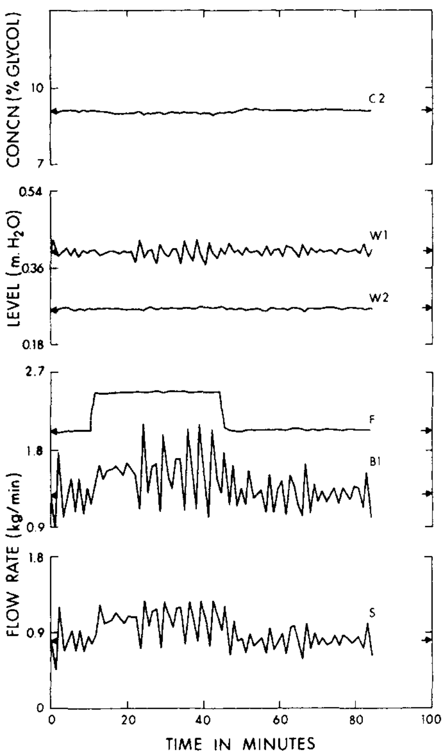

Fig. 4. Experimental response to two 20% changes in feedflow using the optimal feedback-plus-feedforward control (case 4).

level). Although due to an instrument fault, these results are a good illustration of what can occur when high gain feedback control is used with noisy measurements to manipulate control variables.

Feedforward control matrices were also calculated using the zero offset technique (case 1) and by minimization of offsets (case 3) and evaluated for a 20% feed flow rate change. A comparison of Figure 7 versus Figure 4 shows that the experimental responses are very similar and no significant difference in the performance of the three control matrices is apparent from the experimental results.

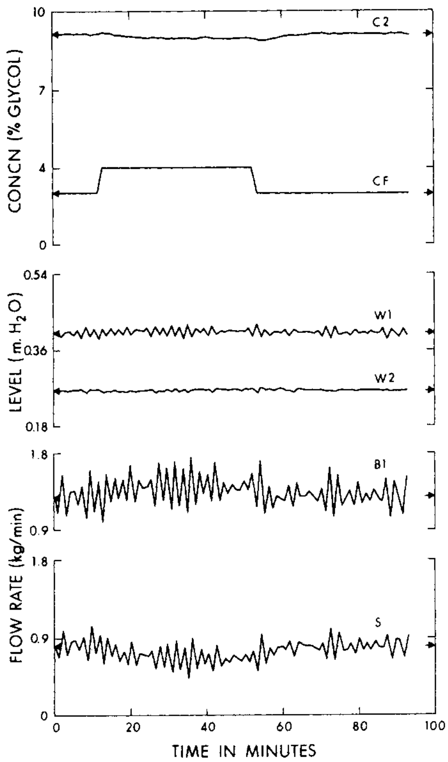

Fig. 5. Experimental response to two 30% changes in feed composition using the optimal feedback-plus-feedforward control (case 4).

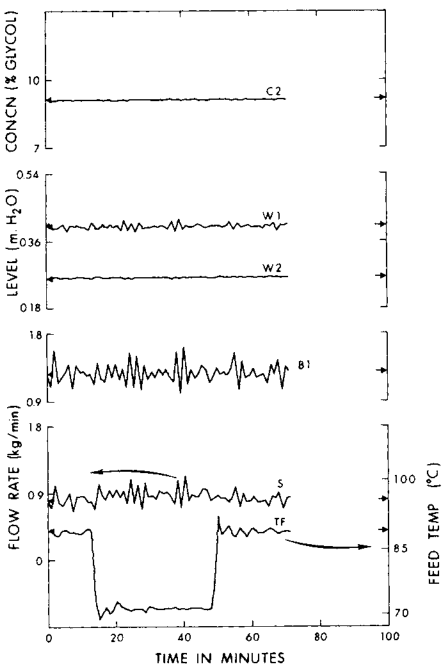

Fig. 6. Experimental response to two 16% changes in feed temperature, using the optimal feedback-plus-feedforward controller (case 4).

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NOTATION

$\mathbf{A}$ = constant coefficient matrix, Equation (2)
$\mathbf{B}$ = constant coefficient matrix, Equation (2)
$\mathbf{B}_1, \mathbf{B}_2$ = partitions of $\mathbf{B}$
$\mathbf{C}$ = constant coefficient matrix, Equation (2)
CC = concentration controller
CR = concentration recorder
$\mathbf{D}$ = constant coefficient matrix, Equation (2)
$\mathbf{d}$ = load vector, $p \times 1$
$\mathbf{e}$ = error vector, $\mathbf{x} - \mathbf{x}_d$
FB = feedback
FC = flow controller
FF = feedforward
FR = flow recorder
$\mathbf{I}$ = unit matrix
i = time interval counter
J = performance index
$\mathbf{K}, \mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3$ = control matrices
LC = level controller
m = dimension of control vector
N = indicator of final time period

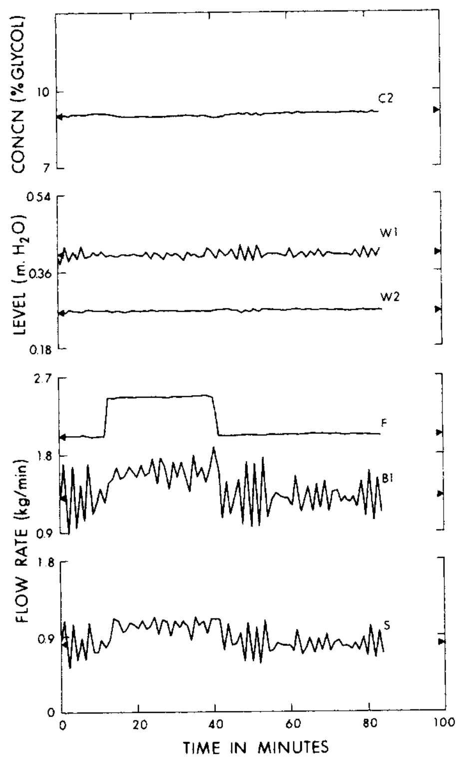

Fig. 7. Experimental response to two 20% disturbances in feedflow using the optimal feedback controller plus feedforward action designed to minimize the final steady state offsets (case 3). Figure 7 is directly comparable with Figures 3 and 4.

n = dimension of state vector
p = dimension of disturbance vector
PC = pressure controller
$\mathbf{Q}$ = state weighting matrix, Equation (23)
q = number of states with zero offset
$\mathbf{R}$ = control weighting matrix, Equation (23)
r = dimension of output vector
$\mathbf{S}$ = final state weighting matrix, Equation (23)
$\mathbf{T}$ = closed loop system matrix defined in Equation (12)
$\mathbf{T}_1, \mathbf{T}_2$ = partitions of $\mathbf{T}$
t = time
T = control or sampling interval for discrete systems
$\mathbf{u}$ = control vector, $m \times 1$
$\mathbf{u}_1, \mathbf{u}_2$ = partitions of $\mathbf{u}$
$\mathbf{x}$ = state vector, $n \times 1$
$\mathbf{x}_1, \mathbf{x}_2$ = partitions of $\mathbf{x}$
$\mathbf{y}$ = output vector, $r \times 1$

Greek Letters

$\mathbf{\Delta}$ = state equation coefficient matrix
$\boldsymbol{\varphi}$ = state equation coefficient matrix
$\boldsymbol{\varphi}^*$ = intermediate matrix defined in Equation (21)
$\boldsymbol{\theta}$ = state transition coefficient matrix

Subscripts

d = desired value
FF = feedforward
FB = feedback
i = iteration counter
s = steady state

Superscripts

$N - i$ = iteration counter
T = matrix or vector transpose
-1 = matrix inverse

PROCESS VARIABLES (REFERENCE FIGURE 1)

State Vector, x		Normal Steady State Value
$W1$	= first effect holdup	14.6 kg
$C1$	= first effect concentration	4.85%
$H1$	= first effect enthalpy	335 kJ/kg
$W2$	= second effect holdup	15.8 kg
$C2$	= second effect concentration	9.64%
Control Vector, u		
S	= steam	0.86 kg/min
$B1$	= first effect bottoms	1.5 kg/min
$B2$	= second effect bottoms	0.75 kg/min
Disturbance Vector, d		
F	= feed flowrate	2.27 kg/min
CF	= feed concentration	3.0%
HF	= feed enthalpy	382 kJ/kg

Other Process Variables (reference Figure 1)

$O1$ = overhead from first effect
$O2$ = overhead from second effect
$P1$ = pressure in first effect
$P2$ = pressure in second effect
TF = temperature of feed
$T1$ = temperature in first effect
$T2$ = temperature in second effect

Recursive Relationships for Controller Constants, Equation (24)

$$\mathbf{K}_{FB}{}^{N-i} = -(\mathbf{\Delta}^T\,\mathbf{P}_{i-1}\,\mathbf{\Delta} + \mathbf{R})^{-1}\,\mathbf{\Delta}^T\,\mathbf{P}_{i-1}\,\boldsymbol{\varphi}$$

$$\mathbf{K}_{FF}{}^{N-i} = -(\mathbf{\Delta}^T\,\mathbf{P}_{i-1}\,\mathbf{\Delta} + \mathbf{R})^{-1}\,(\mathbf{\Delta}^T\,\mathbf{P}_{i-1}\,\boldsymbol{\theta} + \mathbf{\Delta}^T\,\mathbf{W}_{i-1})$$

where

$$\mathbf{P}_{i-1} = \mathbf{T}^T{}_{N-i+1}\,\mathbf{P}_{i-2}\,\mathbf{T}_{N-i+1} + (\mathbf{K}_{FB}{}^{N-i+1})^T\,\mathbf{R}\,\mathbf{K}_{FB}{}^{N-i+1} + \mathbf{Q}$$

$$\mathbf{W}_{i-1} = \mathbf{T}^T{}_{N-i+1}\,\mathbf{W}_{i-2} + \mathbf{T}^T{}_{N-i+1}\,\mathbf{P}_{i-2}\,(\mathbf{\Delta}\,\mathbf{K}_{FF}{}^{N-i+1} + \boldsymbol{\theta}) + (\mathbf{K}_{FB}{}^{N-i+1})^T\,\mathbf{R}\,\mathbf{K}_{FF}{}^{N-i+1}$$

$$\mathbf{T}_{N-i+1} = \boldsymbol{\varphi} + \mathbf{\Delta}\,\mathbf{K}_{FB}{}^{N-i+1}$$

with initial conditions

$$\mathbf{P}_0 = \mathbf{Q} + \mathbf{S}$$

$$\mathbf{W}_0 = \mathbf{0}$$

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The Mixing of Solid Particles in a Motionless Mixer—A Stochastic Approach

A Markov chain model was used to model the axial mixing of solid particles in a motionless mixer having no moving parts. One step transition probabilities were determined experimentally for the model. Based on these transition probabilities, the model was able to predict spatial distribution of tracer particles up to seven steps of the Markov chain, which was equivalent to seven consecutive passes of the mixture through the mixer. Experimental results were in good agreement with those predicted from the Markov chain model.

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SCOPE

Solids mixing, or solids blending, is an operation by which two or more solid materials in particulate form are scattered randomly by the movement of particles in a mixer. The random movement of particles is caused by the motion of the mixer when energy is supplied to the mixer externally or by effect of gravity on the particles. Such operations are common in many industrial processes and can be easily visualized but are difficult to describe quantitatively. Empirical approaches often dominate the design and operation of a mixer. Most of the research on solids mixing has centered around the application of statistics to the mixing problem (Fan et al., 1970; Butters, 1970). Concentration profiles of constituent particles after mixing are difficult to obtain. Therefore, statistical sampling techniques are usually employed to obtain information on the final mixture. Such information is often incomplete because of bias involved in sampling.

Recently the theory of stochastic processes has been applied to analyze and to understand the mixing of solid particles. Models have been proposed to simulate the mixing of particles in some mixers to predict the concentration profiles of constituents of the mixture. Oyama and Ayaki (1956) proposed a Markov chain model to describe the mixing of solid particles in a drum mixer but did not conduct experiments to verify the model. Inoue and Yamaguchi (1969) and Yamaguchi (1969) proposed models of Markov chains to describe solids mixing in a two dimensional V-mixer and in a pan mixer. Mixing experiments were conducted to obtain transition probabilities using a single particle tracing technique. Transition probabilities were difficult to obtain experimentally. However, the concentration profiles of tracer particles predicted by the model based on the experimentally determined transition probabilities were in good agreement with the measured profiles. Extension and application of the results of the two dimensional model to the actual three dimensional mixer, however, is open to question. Oleniczak (1962) postulated a Poisson process for interchange of particles between a volume element and the rest of the mixture. He obtained a stochastic model for the V-mixer. The distribution of tracer particles was found to be bimodal at a low number of revolutions. As the number of revolutions was increased to 30, the bimodal distribution approached the normal distribution. The model did not apply when the number of revolutions exceeded 30.

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